MathCounts Competition (2009)

State Competition Sprint Round
1. A bookcase has 3 shelves with a total of 24 books. The top shelf has 8 mystery books. The middle shelf has 10 math books. The bottom shelf has 6 science books. Two books are now taken off each shelf. What fraction of the books remaining on the three shelves are math books? Express your answer as a common fraction.

2. The Golden Chicken Egg Farm packages eggs in cartons that hold 12 eggs each. What is the smallest number of eggs greater than 10,000 that will fill an integer number of these cartons without any eggs left over?

3. Wilhelm has seven tokens, each with a prime number written on its top face. He notices that these seven numbers are distinct consecutive primes. What is the least possible sum of the prime numbers written on Wilhelm’s seven tokens?

4. Let $\psi$ be the relation defined by $A \psi B = 2A + 5B$. What is the value of $9 \psi (3 \psi 1)$?

5. A flock of geese is swimming on a peaceful lake when a noisy motorcycle drives past causing $\frac{2}{5}$ of the geese to fly away. A little later a small herd of deer runs past and causes $\frac{1}{3}$ of the remaining geese to fly off. Then when a loud siren sounds, another 24 geese leave the lake. Now only 46 geese remain on the lake. How many geese were there in the flock at the start?

6. The probability it will rain on Saturday is 60%, and the probability it will rain on Sunday is 25%. If the probability of rain on a given day is independent of the weather on any other day, what is the probability it will rain on both days, expressed as a percent?
7. Five termites are eating through a piece of wood, all beginning at the same edge and going in the same direction. Woody is 20 mm ahead of Muncher, Cruncher is 10 mm behind Woody, Muncher is 5 mm behind Nibbler, and Biter is 15 mm ahead of Cruncher. How many millimeters is the distance between the two termites that are the farthest apart?

8. Four primes $a, b, c$ and $d$ form an increasing arithmetic sequence with $a > 5$ and common difference 6. What is the ones digit of $d$?

9. On planet Larky, 7 ligs = 4 lags, and 9 lags = 20 lugs. How many ligs are equivalent to 80 lugs?

10. A tank is to be filled with water. When the tank is one-sixth full, 130 gallons of water are added, making the tank three-fifths full. How many gallons does the tank contain when it is completely full?

11. Thirty students took a test on which it was possible to earn only integer scores ranging from 3 to 10, inclusive. If exactly 24 students passed, by earning a score of 7 or higher, what is the highest possible average of the 30 scores? Express your answer as a decimal to the nearest tenth.

12. The mean of five positive integers is 1.5 times their median. Four of the integers are 8, 18, 36 and 62, and the largest integer is not 62. What is the largest integer?
13. In the diagram shown, $\overline{OA} \perp \overline{OC}$ and $\overline{OB} \perp \overline{OD}$. If $m\angle AOD$ is 3.5 times $m\angle BOC$, what is $m\angle AOD$?

14. If three standard, six-faced dice are rolled, what is the probability that the sum of the three numbers rolled is 9? Express your answer as a common fraction.

15. Alfred, Brandon and Charles are the three participants in a race. In how many different ways can the three finish if it is possible for two or more participants to finish in a tie?

16. A rectangular garden has a length that is twice its width. The dimensions are increased so that the perimeter is doubled and the new shape is a square with an area of 3600 square feet. What was the area of the original garden, in square feet?

17. What is the ordered pair $(x, y)$ where $x$ and $y$ are integers, and $2^x - 2^y = 8$?

18. Given that the diagonals of a rhombus are always perpendicular bisectors of each other, what is the area of a rhombus with side length $\sqrt{89}$ units and diagonals that differ by 6 units?
19. If the expression $(12 - x) / 3x$ represents a non-negative integer, what is the largest possible integer value of $x$?

20. Twenty-four 4-inch wide square posts are evenly spaced with 5 feet between adjacent posts to enclose a square field, as shown. What is the outer perimeter, in feet, of the field? Express your answer as a mixed number.

21. What is the smallest solution of the equation $x^4 - 34x^2 + 225 = 0$?

22. The arithmetic progressions {2, 5, 8, 11 ...} and {3, 10, 17, 24 ...} have some common values. What is the largest value less than 500 that they have in common?

23. An isosceles right triangle is removed from each corner of a square piece of paper, as shown, to create a rectangle. If $AB = 12$ units, what is the combined area of the four removed triangles, in square units?

24. If $1 \leq a \leq 10$ and $1 \leq b \leq 36$, for how many ordered pairs of integers $(a, b)$ is $\sqrt{a} + \sqrt{b}$ an integer?
25. Six students are being grouped into three pairs to work on a science lab. How many different combinations of three pairs are possible?

26. In trapezoid $ABCD$, $AB$ is parallel to $CD$, $AB = 7$ units, and $CD = 10$ units. Segment $EF$ is drawn parallel to $AB$ with $E$ lying on $AD$ and $F$ lying on $BC$. If $BF:FC = 3:4$, what is $EF$? Express your answer as a common fraction.

27. An integer $X$ has the following properties:

1.) $X$ is a multiple of 17
2.) $X$ is less than 1000
3.) $X$ is one less than a multiple of 8.

What is the largest possible value of $X$?

28. Points $X$, $Y$ and $Z$ lie on the sides of triangle $ABC$ so that segments $AX$, $BY$ and $CZ$, if drawn, would intersect at one interior point $P$. Using 3 of these 7 points at a time as vertices, how many triangles can be formed?

29. The sum of $n$ positive integers, not necessarily distinct, is 22. What is the largest possible product of the $n$ integers?

30. In the figure, point $A$ is the center of the circle, the measure of angle $RAS$ is 74 degrees, and the measure of angle $RTB$ is 28 degrees. What is the measure of minor arc $BR$, in degrees?